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A Study of Poly-Basic Hypergeometric Functions And Their Applications

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Abstract
It's becoming more common for hypergeometric functions to be used in fields such as Statistics. There is an introduction to the q-hypergeometric series, or basic hypergeometric series, in the course notes, as well as some basic summation and transformation findings. An attempt has been made to create equations for basic and polybasic hypergeometric series using a known identity and summation formulae for truncated q-series.

Keyword
summation formula, transformation formula, basic hypergeometric series, poly-basic hypergeometric series.

Introduction
In mathematical analysis and its applications, the hypergeometric function $2F1(a, b; c; z)$ is critical. These problems include conformal mapping of triangular domains bounded by line segments or circular arcs, as well as many quantum physics problems. This function allows us to solve these and many more. The majority of the functions encountered in the investigation are hypergeometric function variants. A series can be summed into an elementary function when a hypergeometric series is first introduced and investigated by Gauss. It is possible to use the contiguous function relations to evaluate the contiguous functions of a hypergeometric function, as well as to derive the summation and transformation formulas for such series. Differential equations with a second-order hypergeometric coefficient can alternatively be expressed as solutions to hypergeometric functions. Hypergeometric series evaluation, derivation of summation and transformation formulas for such series, and evaluation of the contiguous functions to a hypergeometric function are all applications of the contiguous function relations.
The advent of more powerful computer hardware has had a profound impact on how numerical computations are done. Iterative processes in numerical methods include making a series of approximations and then hoping that the process will eventually lead to the correct solution. The use of computer-oriented numerical approaches has permeated every aspect of modern scientists' work and research. There has been a remarkable gain in numerical computing efficiency and speed due to the development of ultra-powerful computers.

Application
Applications of a Bailey-type Transformation
If k is set equal to a_q in the definition of a WP Bailey pair,

$$\beta_n(a,k) = \sum_{j=0}^{n} \frac{(k/a)_{n-j}\,(k)_{n+j}}{(q)_{n-j}\,(aq)_{n+j}}\, a_j(a,k)\,,$$

This equation becomes $\beta_n = \sum_{j=0}^{n} a_j$ when simplified. Despite its simplicity, the connection between the α_n's and the β_n's has several interesting ramifications, such as several new basic hypergeometric summation formulae, a connection to the Prouhet-Tarry-Escott problem, and new identities of the Rogers–Ramanujan–Slater type.

Review of Literature

A wide number of extremely interesting results contiguous with the Gauss second, Kummer, and Bailey theorems for the series $2F1$ have been obtained by Lavoie et al. Hypergeometric series and their contiguous relationships are discussed in length in this chapter.

Many contributions were made by R. P. Agrawal [Agrawal (1967)], [Agrawal (1976)], and [Agrawal (1981)]. For his dissertation, he worked on fractional Q derivatives, q-integrals, and mock theta functions. He also did combinatorial analysis and extended Meijer's G Function, Pade approximants, and continuing fractions. Quadratic transformations of fundamental series were studied by W.A. Al-Salam and A. Verma (Al-Salam et al., 1972). Not at all.

In 2012, Mubeen and Habibullah [25] defined k-fractional integration and gave its applications regarding fractional integrals. In 2012, Mubeen and Habibullah [26] also gave a useful and simple integral representation of some confluent k-hypergeometric functions $1F1$, and k-hypergeometric functions $2F1,k$. Furthermore, in 2013, Mubeen [27] defined a second order linear k-hypergeometric differential equation $k(1 - kz)\omega'' + [c - (a + b + k)kz]\omega' - ab\omega = 0$ having one solution in the form of k-hypergeometric function $2F1,k(a, b; c; z)$.

He studied the generalised q hypergeometric function and continuing fractions under Bhagirathi. To study the transformations of basic hypergeometric series without termating, V. K. Jain and M. Verma used their contour integrals and applications to Ramanujan identities.

When it comes to basic hypergeometric functions, one name stands out: H. S. Shukla [Shukla (1993)]. They worked on summation formulas for qhypergeometric series, summation formulas for non-terminating basic hypergeometric series in, q analogue of a transformation of Whipple, and transformations between basic hypergeometric series on various bases and identities of RogersRamanujan Type. BD Sears [Sears (1951)] worked on basic hypergeometric function transformation theory. There has been research into the identities of Roger Ramanujan types done by the late P. Rastogi [see Rastogi (1984)]. They studied the transformations of the basic bilateral series' products and their transformations by A.Verma and M. Upadhyay (Verma et al., 1968)).

At that time, Morita calculated the hypergeometric function $2F1[n + 1/2, n + 1/2; m; z]$ using Gauss contiguous relations. Hypergeometric functions and orthogonal polynomials were discovered in 1996 by Gupta et al. [12]. In 2002, Vidunas used contiguous relations to expand the Kummer identity. There have been numerous attempts to compute these generic contiguous connections, but Vidunas was the first to establish several features of the coefficients in 2003 and offer efficient methods for doing so. When Rakha and Ibrahim [15] studied $2F1$'s contiguous relations in 2006, they discovered some surprising results. The contiguous relations and their computations for $2F1$ hypergeometric series were derived in 2008 by Ibrahim and Rakha [16]. As a linear relationship between three shift Gauss polynomials in the parameters a, b, and c, they discovered the intriguing formula they used in their research.

There have been recent publications by Denis, Denis and Singh and Singh that describe the establishment of poly-basic hypergeometric series transformations. After that, they were able to derive sums and multiplications for poly-basic hypergeometric functions, which led to some fascinating transformations.. The findings presented here support a reorientation of the q-series (basic hypergeometric series) theory.

Research Methodology

In 1947, the Proceeding of the London Mathematical Society published Bailey's significant contribution to hypergeometric and fundamental hypergeometric series, particularly. The following is a significant finding from the study that was later dubbed Bailley's transformation:

$$if\ \beta_n = \sum_{r=0}^{n} a_r u_{n-r} v_{n+r},$$

$$\gamma_n = \sum_{r=n}^{\infty} \delta_r u_{r-n} v_{n+r},$$

where α_r , δ_r , u_r , v_r are functions of r only, such that the series for γ_n exists.

Result and Discussion

THE q-BINOMIAL THEOREM, THE q-GAMMA AND q-BETA FUNCTIONS.

The q-binomial formula is the most basic summation formula in the theory of fundamental hypergeometric series:

$1\Phi 0[a; -; q; z]$

$$= \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty}, |z| < 1 . \quad (1)$$

Heine's proof of (1) was based on the observation that both sides of (1) satisfy the fundamental equation

$$(1-z)f(z) = (1-az)f(az), \quad (2)$$

which, together with f(0) = 1, determines a unique analytic function f(z) inside the unitcircle.

For (0)<q<l, the q-gamma function is defined by

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, Re\ x > 0 , \quad (3)$$

was first introduced by Thomae [1] and later by Jackson

Also $\lim q \to 1, \Gamma_q[x] = \Gamma(x)$. (4)

In view of [3] it is natural to define the q-beta function by

$$B_q(x,y) = \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)} ,0 < q < 1, Re\ x, y > 0. \quad (5)$$

By [1.2(A).3] and [1.2(A).5] we have

$$B_q(x,y) = (1-q)\frac{(q, q^{x+y}; q)_\infty}{(q^x, q^y; q)_\infty}$$

$$= (1-q)\frac{(q;q)_\infty}{(q^y;q)_\infty}\sum_{n=0}^{\infty}\frac{(q^y;q)_n}{(q;q)_n}q^{nx}$$

$$= (1-q)\sum_{n=0}^{\infty}\frac{(q^{n+1};q)_\infty}{(q^{n+y};q)_\infty}q^{nx}$$

Thomae and Jackson introduced the q-integral

$$\int_0^1 f(t)d_q t = (1-q)\sum_{n=0}^{\infty} f(q^n)q^n, \quad (6)$$

which was generalized by Jackson to

$$\int_a^b f(t)d_qt = \int_0^b f(t)d_q\, t - \int_0^a f(t)d_q\, t \qquad (7)$$

With

$$\int_0^a f(t)d_qt = a(1-q)\sum_{n=0}^{\infty} f(aq^n)q^n, \qquad (8)$$

and

$$\int_0^{\infty} f(t)d_qt = (1-q)\sum_{n=-\infty}^{\infty} f(q^n)q^n, \qquad (9)$$

Transformations of poly-basic hypergeometric functions

We shall establish certain transformation of basic hypergeometric functions with more than two bases.

(A) $$\sum_{k=0}^{n} a_k \sum_{j=0}^{n-k} A_j = \sum_{k=0}^{n} A_k \sum_{j=0}^{n-k} a_j$$

(B) $$2\Phi1\begin{bmatrix} a, y; q; q \\ ayq \end{bmatrix}_N = \frac{[aq, yq; q]_N}{[q, ayq; q]_N}.$$

(C) $$4\Phi3\begin{bmatrix} a, q\sqrt{a}, -q\sqrt{a}, e; q; 1/e \\ \sqrt{a}, -\sqrt{a}, aq/e \end{bmatrix}_N = \frac{[aq, eq; q]_N}{[q, aq/e; q]_N e^N}.$$

$$6\Phi5\begin{bmatrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, d & ; q; q \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d & \end{bmatrix}_N = \frac{[aq, bq, cq, dq; q]_N}{[q, aq/b, aq/c, aq/d; q]_N},$$

(D) With a=bcd.

(E) $$\sum_{k=0}^{n} \frac{(1-ap^kq^k)[a;p]_k[c;q]_k c^{-k}}{(1-a)[q;q]_k[ap/c;p]_k} = \frac{[ap;p]_n[cq;q]_n c^{-n}}{[q;q]_n[ap/c;p]_n}$$

$$\sum_{k=0}^{n} \frac{(1-ap^kq^k)(1-bp^kq^{-k})[a;b;p]_k[c,a/bc;q]_k q^k}{(1-a)(1-b)[q;aq/b;q]_k[ap/c, bcp;p]_k}$$

(F) $$= \frac{[ap;bp;p]_n[cq;aq/bc;q]_n}{[q;aq/b;q]_n[ap/c, bcp;p]_n}.$$

If we take

$$a_k = \frac{(1-ap^kq^k)[a;p]_k[b;q]_k b^{-k}}{(1-a)[q;q]_k[ap/b;p]_k}$$

And

$$A_k = \frac{[c, q\sqrt{c}, -q\sqrt{c}, d; q]_k}{[q_1, \sqrt{c}, -\sqrt{c}, cq_1/d; q_1]_k d^k}$$

in (A) and use (C) and (E), we get

$$5\Phi4\left[\begin{matrix} b: apq; a, dq_1^{-n}/c, q_1^{-n}; q, pq, p, q_1; 1/bd \\ -: a: ap/b; q_1^{-n}/c, q_1^{-n}/d \end{matrix}\right]$$

$$= \frac{[bq,q]_n[ap,p]_n[q_1,cq_1/d;q_1]_n(d/b)^n}{[q;q]_n[ap/b,p]_n[cq_1,dq_1;q_1]_n}$$

$$\times\ 6\Phi5\left[\begin{matrix} c, q_1\sqrt{c}, -q\sqrt{c}, d: bp^{-n}/a; q^{-n}; q_1, p, q; 1/bd; 1/bd \\ \sqrt{c}, -\sqrt{c}, cq_1/d: p^{-n}/a; q^{-n}/b \end{matrix}\right].$$

(ii) Again, if we set

$$a_k = \frac{(1-ap^kq^k)[a;p]_k[b;q]_kb^{-k}}{(1-a)[q;q]_k[ap/b;p]_k}$$

And

$$A_k = \frac{[c,d;q_1]_k[b;q]_kq_1^k}{[cdq_1;q_1]_k[q_1;q_1]_k}$$

in (A) and use (B) and (E), we get

$$5\Phi4\left[\begin{matrix} b: apq; a, q_1^{-n}/cd, q_1^{-n}; q, pq, p, q_1; 1/b \\ -: a: ap/b; q_1^{-n}/c, q_1^{-n}/d \end{matrix}\right]$$

$$= \frac{[ap,p]_n[bp,q]_n[cdq_1,q_1;q_1]_n}{[ap/b;p]_n[q;q]_n[cq_1,dq_1;q_1]_n}b^n \times$$

$$\times\ 4\Phi3\left[\begin{matrix} c, d: bp^{-n}/a; q^{-n}; q_1, p, q; q_1/b \\ cdq_1: p^{-n}/a; q^{-n}/b \end{matrix}\right].$$

(iii) Further, taking

$$a_k = \frac{(1-ap^kq^k)(1-bp^kq^k)[a,b;p]_k[c,a/bc;q]_kq^k}{(1-a)(1-b)[q,aq/b;q]_k[ap/c,bcp;p]_k}$$

And

$$A_k = \frac{[c,q_1\sqrt{d},-q_1\sqrt{d},e;q_1]_k}{[q_1,\sqrt{d},-\sqrt{d},dq_1/e;q_1]_ke^k}$$

in (A) and using (C) and (F), we get

$$8\Phi7\left[\begin{matrix} c, a/bc: apq; bp/q: \quad a, b: eq_1^{-n}/d, q_1^{-n}; q, pq, p/q, p, q_1; q/e \\ aq/b: \quad a; b \qquad\qquad : ap/c, bcp: q_1^{-n}/d, q_1^{-n}/e \end{matrix}\right]$$

$$= \frac{[ap,bp;p]_n[cq,aq/bc;q]_n[q_1,dq_1/e;q_1]_ne^n}{[ap/c,bcp;p]_n[q,aq/b;q]_n[dq_1;eq_1;q_1]_n} \times$$

$$\times\ 8\Phi7\left[\begin{matrix} d, q_1\sqrt{d}, -q_1\sqrt{d}, e: q^{-n}, bq^{-n}/a: cp^{-n}/a, p^{-n}/bc; q_1, q, p; 1/e \\ \sqrt{d}, -\sqrt{d}, dq_1/e: q^{-n}/c, bcq^{-n}/a: p^{-n}/a, p^{-n}/b \end{matrix}\right]$$

(iv) Next, taking

$$a_k = \frac{(1-ap^kq^k)[a;p]_k[b;q]_kb^{-k}}{(1-a)[q;q]_k[cp/b;p]_k}$$

And

$$A_k = \frac{[c,q_1\sqrt{c},-q_1\sqrt{c},d,e,f;q_1]_kq_1^k}{[q_1,\sqrt{c},-\sqrt{c},cq_1/d,cq_1/e,cq_1/f;q_1]_ke^k}, (c = def)$$

in (A) and making use of(D) and (E), we get

$$7\Phi6\left[\begin{matrix} b: apq; a: dq_1^{-n}/c, eq_1^{-n}/c, fq_1^{-n}/c, q_1^{-n}; q, pq, p, q_1; 1/b \\ -: a: ap/b: q_1^{-n}/c, q_1^{-n}/d, q_1^{-n}/e, q_1^{-n}/f \end{matrix}\right]$$

$$= \frac{[ap;q]_n[bq;q]_n[q_1,cq_1/d,dq_1/e,dq_1/f;q_1]_n}{[ap/b;p]_n[q;q]_n[cq_1,dq_1,eq_1,fq_1;q_1]_nb^n} \times$$

$$\times\, 8\Phi7\begin{bmatrix} c, q_1\sqrt{c}, -q_1\sqrt{c}, d, e, f : q^{-n} : bq^{-n}/a\, ; q_1 q, p; q_1\, /\, b \\ \sqrt{c}, -\sqrt{c}, cq_1/d\, , cq_1/e\, , cq_1/f : q^{-n}/b : p^{-n}\, /\, a \end{bmatrix}$$

(v) Again, if we set

$$a_k = \frac{(1-ap^k q^k)(1-bp^k q^k)[a,b;p]_k[c,a\,/\,bc;q]_k q^k}{(1-a)(1-b)[q,aq\,/\,b;q]_k[ap/c,bcp\,;p]_k}$$

And

$$A_k = \frac{[d,q_1\sqrt{d},-q_1\sqrt{d},e,f,g;q_1]_k q_1^k}{[\sqrt{d},-\sqrt{d},dq_1/e,dq_1/f\,,dq_1/g\,;q_1]_k}\,, \quad d = efg$$

in (A) and use (D) and (F), we get

$$10\Phi9\begin{bmatrix} c, a/bc : apq; bp/q : a, b : eq_1^{-n}/d\, , fq_1^{-n}/d\, , gq_1^{-n}/d\, , q_1^{-n}; q, pq, p/q\, ,, p, q_1; q \\ aq/b : a, b : ap/c\, , bcp : q_1^{-n}/d\, , q_1^{-n}/e\, , q_1^{-n}/f\, , q_1^{-n}\, /\, g \end{bmatrix}$$

$$= \frac{[ap,bp;q]_n[cq,aq\,/\,bc;q]_n[q_1,dq_1/e\,,dq_1/f\,,dq_1/g\,;q_1]_n}{[q,aq\,/\,b;q]_n[ap/c\,,bcp;p]_n[dq_1,eq_1,fq_1,gq_1;q_1]_n}\times$$

$$\times\, 10\Phi9\begin{bmatrix} d, q_1\sqrt{d}, -q_1\sqrt{d}, e, f, g : q^{-n}, bq^{-n}/a : cq^{-n}/a\, , p^{-n}/bc\, ; q_1, q, p; q_1 \\ \sqrt{d}, -\sqrt{d}, dq_1/e\, , dq_1/f\, , dq_1/g : q^{-n}/c\, , bcq^{-n}/a : p^{-n}/a\, , p^{-n}\, /\, b \end{bmatrix}$$

Conclusion

The q technique can be used in a wide variety of situations. For example, it's employed in subjects like solid state theory, mechanical engineering and operational calculus, as well as in quantum theory, cosmology, Lie theory and linear algebra.

References

1. Agarwal, R.P., Fractional q-derivative and q-integrals and certain hypergeometric transformation, Ganita, Vol. 27 25-32,1976.
2. Agarwal, R.P., An extension of Meijer's G-function, proc. Nat. Inst. Sci (India),Vol (31) A, 536-46, 1965.[3]. Agarwal, R.P., Certain basic hypergeometric identities associated with mock theta functions, Quart. J. Math. (Oxford) 20 (1968), 121-128, 1968.
3. Shukla, H.S., Certain Transformation in the Field of Basic Hypergeometric Functions, Ph.D. Thesis, Purvanchal,Jaunpur, 1993.
4. Bhagirathi, N. A., On certain investigations in q-series and continued fractions, Math. Student, 56, 158 – 170,1988.
5. Al-Salam, W.A. and Chihara, T.S.,Another characterization of the classical orthogonal polynomials, SIAM Journal of Mathematical Analysis, 3, 65–70, 1972.
6. Verma, A. and Jain, V.K., Some summation formulae of Basic hypergeometric series, Indian J. of Pure and applied Math. 11 (8), 1021-1038,1980.
7. Denis, R.Y., On certain summation of q-series and identities of RogersRamanujan type, J. Math. Phys. Sci., 22 (1) (1988), 87-99.
8. Denis, R.Y., Singh, S.N., Singh, S.P., On certain Transformation and Summation formulae for q-series, Italian Jornal of Pune and Applied Mathematics, 27 (2010), (79-190.
9. T. Morita, "Use of the gauss contiguous relations in computing the hypergeometric functions 2F1(n + 1/2, n + 1/2; m ; z)," Interdisciplinary Information Sciences, vol. 2, no. 1, pp. 63–74, 1996.
10. R. Vidunas, "A generalization of Kummer's identity," The Rocky Mountain Journal of Mathematics, vol. 32, no. 2, pp. 919–936, 2002.

11. R. Vidunas, “Contiguous relations of hypergeometric series,”Journal of Mathematical Analysis and Applications, vol. 135, pp.507–519, 2003
12. M. A. Rakha and A. K. Ibrahim, “On the contiguous relationsof hypergeometric series,” Journal of Computational and AppliedMathematics, vol. 192, no. 2, pp. 396–410, 2006.
13. Verma, A. and Jain, V.K., Some summation formulae of Basic Hypergeometric Series, Indian J. of Pure and applied Math. 11 (8), 1021-1038,1980.
14. "Certain basic hypergeometric identities associated with mock theta functions", Quart. J. Math.(Oxford) 20 (1968), 121-128.
15. Selected topics in special functions; (poly basic hypergeometric series by A. Verma (P.77-92), Allied Publishers limited, New Delhi 2001.
16. Laughlin, J. M., Some identities between basic hypergeometric series deriving from a new Bailey type transformation, The J. of Mathematical Analysis and its Applications, 345(2), (2008), 670-677.
17. Laughlin, J. M., and Zimmer, P., Some applications of bailey type transformations, International Mathematical Forum, 5 (2010), 61, 3007-3022.
18. Mishra, B. P., On Certain transformation formulae involving Basic Hypergeometric Function, Journal of Ramanujan Society of Mathematics and Mathematical Sciences, Vol. 2, No.2, (2014), 09-16.
19. Yadav, R. K. and Mishra P. K., Some new transformations on basic hypergeometric series using Bailey’s transform, Advances in Applied Science and Research, 5(6), (2014), 97-102.
20. R. D´ıaz, C. Ortiz, and E. Pariguan, “On the *k*-gamma *q*-distribution,” Central European Journal of Mathematics, vol. 8, no. 3,pp. 448–458, 2010.
21. S. Mubeen and G. M. Habibullah, “*k*-fractional integrals and application,” International Journal of Contemporary Mathematical Sciences, vol. 7, no. 1–4, pp. 89–94, 2012.
22. S. Mubeen and G. M. Habibullah, “An integral representation of some *k*-hypergeometric functions,” International Mathematical Forum, vol. 7, no. 1–4, pp. 203–207, 2012.
23. S. Mubeen, “Solution of some integral equations involving conuent k-hypergeometric functions,” Journal of Applied Mathematics, vol. 4, no. 7, pp. 9–11, 2013.